Recall the basic rules for exponents:

(i) $x^1 = x$

(ii) $x^m \cdot x^n = x^{m+n}$

(iii) $(x^m)^n = x^{mn}$

(iv) $\dfrac{x^m}{x^n} = x^{m-n} = \dfrac{1}{x^{n-m}}$

(v) $x^{-n} = \dfrac{1}{x^n}$

(vi) $x^0 = 1$

(vii) $a^n b^n = (ab)^n$

Def. An exponential function with base b is the function of the form $f(x) = b^x$ where b is a real number greater than 0 $(b > 0)$.

Examples

Let $g(x) = \left(\frac{1}{4}\right)^x$ and $h(x) = 10^{x-2}$. Then

(a) $g\left(-\frac{3}{2}\right) = \left(\frac{1}{4}\right)^{-3/2} = \dfrac{1^{-3/2}}{4^{-3/2}}$

$= \dfrac{1}{4^{-3/2}}$

$= 4^{3/2}$

$= (\sqrt{4})^3$

$= 2^3$

$= 8.$

(b) $h(2.3)$

$= 10^{2.3-2}$

$= 10^{0.3}$

(c) $g(0)$

$= \left(\frac{1}{4}\right)^0$

$= 1$

Exercise

Let $g(x) = \left(\frac{1}{9}\right)^x$ and $h(x) = 5^{x-2}$. Find

(a) $g\left(-\frac{3}{2}\right)$

(b) $h(2.9)$

Graphs

Graph the function $f(x) = 5^x$

Soln.

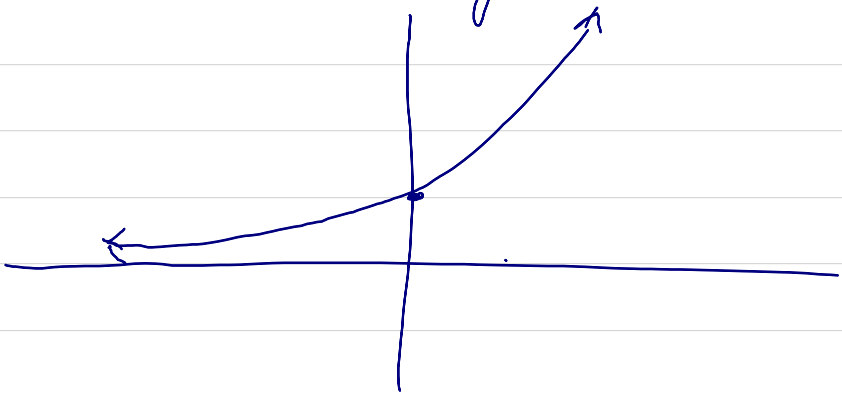
x	y
0	$5^0 = 1$
1	$5^1 = 5$
-1	$5^{-1} = \frac{1}{5}$

What if x is very large positive? 5^x is very large

What if x is very large negative?

$$5^{-1000} = \frac{1}{5^{1000}}$$

$$= \text{very small}$$



Graph $f(x) = \left(\frac{2}{3}\right)^x$

Soln.

x	y
1	$\frac{2}{3}$
0	1
-1	$3/2$

What if x is very large positive?

$$\left(\frac{2}{3}\right)^{1000} = \underbrace{\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdots \frac{2}{3}}_{1000 \text{ times}}$$

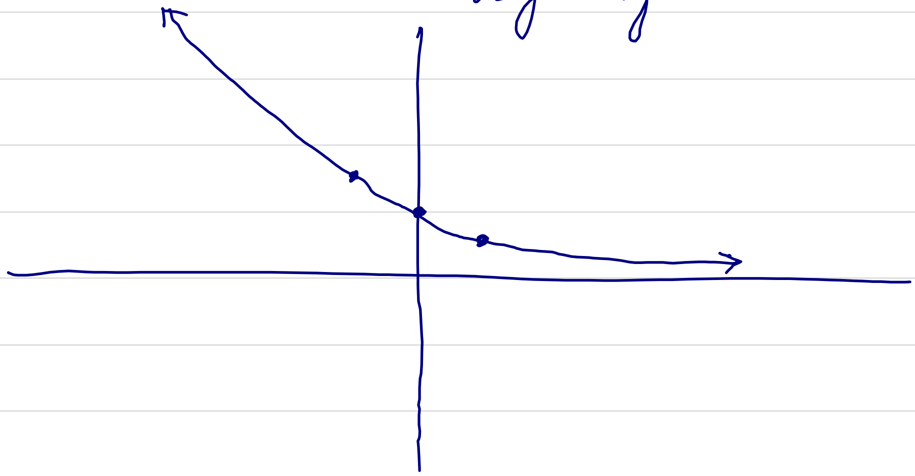
$$= \text{very small}$$

What if x is very large negative?

$$\left(\frac{2}{3}\right)^{-1000} = \frac{1}{\left(2/3\right)^{1000}} = \frac{1}{\frac{2^{1000}}{3^{1000}}} = \frac{3^{1000}}{2^{1000}} = \left(\frac{3}{2}\right)^{1000}$$

$$= \underbrace{\frac{3}{2} \cdot \frac{3}{2} \cdot \frac{3}{2} \cdots \frac{3}{2}}_{1000 \text{ times}}$$

$$= \text{very large.}$$



Graph $f(x) = 2^{x-1} + 1$

Soln. We will use transformations which we studied in Chapter 1.

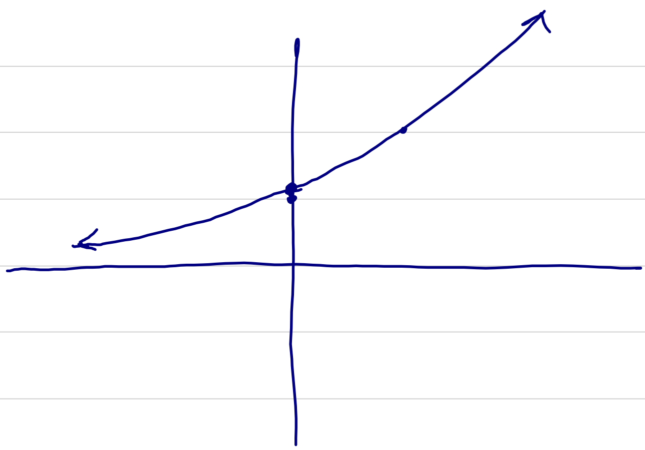
Observe that the main function is $M(x) = 2^x$.

$$M(x) \xrightarrow{\text{shift right}} M(x-1) \xrightarrow{\text{shift up}} M(x-1)+1$$

$$2^x \longrightarrow 2^{x-1} \longrightarrow 2^{x-1}+1$$

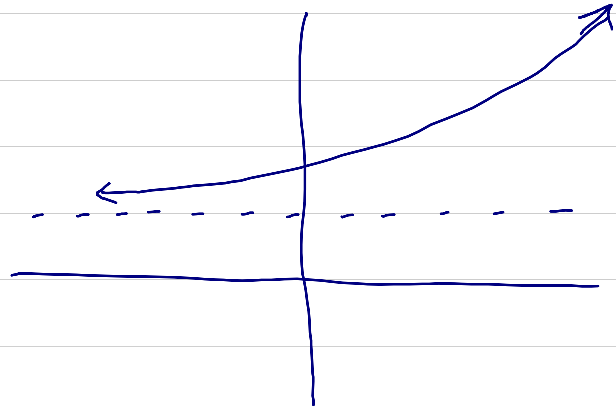
Main function: 2^x

x	y
0	1
1	2
-1	$\frac{1}{2}$



Shift right $\longrightarrow$

Shift up $\longrightarrow$



Exercises

Graph $3^{x+1}-2$

$2^{x+1}+3$.

Compound Interest Formula

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

P is principal / initial amount

r is rate of interest (Note this is given in percentage).

n is the number of times compounded per year

t is the number of years

A is the final amount.

Ex. Say you open a savings account with \$100. The bank gives you compound interest of 6% semiannually. How much will you have in the bank in 1 year?

Soln. Before we use the formula let's do base bones calculation. This will give you an idea on why the formula works. In fact it's not difficult to derive the general formula.

Since it is compounded semiannually, the bank gives you $\frac{6}{2}\% = 3\%$ every six months.

Time	Balance
Year 0	100
6 months.	$100 + \frac{3}{100} \cdot 100$
Year 1	$100 + \frac{3}{100} \cdot 100 + \frac{3}{100}\left(100 + \frac{3}{100} \cdot 100\right)$

$$= \left(100 + \frac{3}{100} \cdot 100\right)\left(1 + \frac{3}{100}\right)$$

$$= 100\left(1 + \frac{3}{100}\right)\left(1 + \frac{3}{100}\right)$$

$$= 100\left(1 + \frac{3}{100}\right)^{2}$$

$$= 100\left(1 + \frac{6}{2 \cdot 100}\right)^{2 \cdot 1} = 100\left(1 + \frac{0.06}{2}\right)^{2 \cdot 1}$$

Using formula:

$$A = P\left(1 + \frac{r}{2}\right)^{nt}$$

$$= 100\left(1 + \frac{0.06}{2}\right)^{2 \cdot 1}$$

$$= 100\,(1.03)^{2}$$

$$= 106.9$$

You will have \$106.9 in account after a year. Note that this is 90¢ more than a flat rate 6% interest. But over time this adds up really fast. Ask Warren Buffett.

Note

Annually	$n=1$
Semiannually	$n=2$
Quarterly	$n=4$
Monthly	$n=12$
Weekly	$n=52$
Daily	$n=365$

Exercise. If \$5000 deposited in an account paying 6% compounded annually, how much will you have in the account in 4 years?

Derivation of Euler's base e

we have

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

$$= P\left(1+\frac{1}{\frac{n}{r}}\right)^{\frac{n}{r}rt}$$

Let $m = \frac{n}{r}$. Then

$$A = P\left(1+\frac{1}{m}\right)^{mrt}$$

$$= \left[P\left(1+\frac{1}{m}\right)^{m}\right]^{rt}$$

Euler's number e is defined as

$$\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m}$$

i.e., if m is very large $\left(1+\frac{1}{m}\right)^m$ is very close to $e \approx 2.71828$.

Thus, we have

$$A = Pe^{rt}$$

when the n in the compound interest formula is very large.

<u>Ques.</u> What does n very large mean?

<u>Ans.</u> Note $n = 1$ annually

$n = 2$ semiannually

$n = 365$ every day

$n =$ very large every nano second.

In other words, if the interest is compounded continuously, then we get the formula

$$A = Pe^{rt}$$

<u>Ex.</u> If \$3000 is deposited in savings account paying 3% a year compounded continuously, how much will you have in the account in 7 years?

<u>Soln.</u>

$$A = Pe^{rt}$$

$$= 3000\, e^{(0.03)\cdot 7}$$

$$\approx 3701.034$$

There will be \$3701.03 in the account in 7 years.